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Moment inequalities for the maximum of partial sums of random fields

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§ 1. Introduction and preliminaries

Let $\{\zeta_{kl}\}$ $(k, l=1, 2, \ldots)$ be a random field. It is not assumed that the random variables (in abbreviation: rv's) ζ_{kl} are mutually independent or identically distributed. Set

$$S(b, m;\ c, n) = \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} \zeta_{kl}$$

and

$$M(b, m;\ c, n) = \max_{1 \leqq p \leqq m} \max_{1 \leqq q \leqq n} |S(b, p;\ c, q)|,$$

where $b, c \geqq 0$, and $m, n \geqq 1$ are integers.

The subject of this paper is to provide bounds on $E\big(M^\gamma(b, m;\ c, n)\big)$ in terms of given bounds on $E|S(b, m;\ c, n)|^\gamma$, where γ is a given positive exponent. We emphasize that the only restrictions on the dependence will be those imposed by the assumed bounds for $E|S(b, m;\ c, n)|^\gamma$. These assumed bounds are guaranteed under a suitable dependence restriction, e. g., martingale difference, multiplicativity of finite order, orthogonality, mixing condition, or the like.

Bounds on $E\big(M^\gamma(b, m;\ c, n)\big)$ are of use in deriving convergence properties of $S(m, n) = S(0, m;\ 0, n)$ as $m, n \to \infty$, probability inequalities for $M(b, m;\ c, n)$, and tightness criteria for certain sequences of random functions (see [3]). To develop such results under various dependence restrictions, the theorems of this paper reduce the problem of placing appropriate bounds on $E\big(M^\gamma(b, m;\ c, n)\big)$ to the much easier problem of placing appropriate bounds on $E|S(b, m;\ c, n)|^\gamma$. Various applications of our theorems, for example to obtain strong laws of large numbers, will be presented in a subsequent paper [8].

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The proofs are based on the "bisection" technique, which goes back to Rademacher and Menšov; see, e.g., BILLINGSLEY [3, pp. 87—103]. The treatment is similar to [6]. The results obtained can be considered as extensions of those in [6] from sequences $\{\xi_j\}$ of rv's to random fields $\{\zeta_{kl}\}$.

In the following, $f(b, m; c, n)$ will denote a non-negative function depending on the joint distribution function (in abbreviation: df) of $\{\zeta_{kl}: k=b+1, \dots, b+m; l=c+1, \dots, c+n\}$, and possessing the following two properties of a rather general nature:

$$f(b, h; c, n)+f(b+h, m-h; c, n) \leqq f(b, m; c, n) \tag{1.1}$$

and

$$f(b, m; c, i)+f(b, m; c+i, n-i) \leqq f(b, m; c, n) \tag{1.2}$$

for all $b \geqq 0$, $1 \leqq h < m$ and $c \geqq 0$, $1 \leqq i < n$. In other words, condition (1.1) means that $f(b, m; c, n)$ as a function of the interval $(b+1, b+m)$ is "superadditive" for fixed c and n, while (1.2) expresses the superadditivity in $(c+1, c+n)$ for fixed b and m. Examples are $f(b, m; c, n)=m^{\beta_1}n^{\beta_2}$ with $\beta_1, \beta_2 \geqq 1$ or $f(b, m; c, n)= = \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} \sigma_{kl}^2$, in the latter case assuming the existence of the finite variances σ_{kl}^2 of the rv's ζ_{kl}.

The upper bound on $E|S(b, m; c, n)|^\gamma$ will be considered in the general form

$$E|S(b, m; c, n)|^\gamma \leqq f^\alpha(b, m; c, n), \tag{1.3}$$

where α and γ are given numbers, $\alpha \geqq 1$, $\gamma > 0$, and $f(b, m; c, n)$ satisfies (1.1)—(1.2).

The treatment of the case $0 < \gamma \leqq 1$ is quite simple. In fact, applying the well-known inequality

$$E|\xi+\eta|^\gamma \leqq E|\xi|^\gamma + E|\eta|^\gamma \quad (0 < \gamma \leqq 1),$$

we obviously have

$$E\big(M^\gamma(b, m; c, n)\big) \leqq \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} E|\zeta_{kl}|^\gamma \leqq \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} f^\alpha(k-1, 1; l-1, 1),$$

provided (1.3) holds for all $b, c \geqq 0$ and $m, n \geqq 1$. Now using (1.1)—(1.2) and the elementary inequality

$$\sum_{i=1}^{r} u_i^\alpha \leqq \Big(\sum_{i=1}^{r} u_i\Big)^\alpha, \quad \text{where} \quad \alpha \geqq 1 \quad \text{and} \quad u_i \geqq 0 \quad (i=1, 2, \dots, r),$$

we arrive at the following result.

Theorem 1. *Suppose that there exists a non-negative function $f(b, m; c, n)$ satisfying* (1.1)—(1.2) *such that* (1.3) *holds for all $b, c \geqq 0$ and $m, n \geqq 1$, where $\alpha \geqq 1$*

and $0<\gamma\leqq 1$. *Then we have*

$$E\big(M^{\gamma}(b, m;\ c, n)\big) \leqq f^{\alpha}(b, m;\ c, n)$$

for all $b, c\geqq 0$ *and* $m, n\geqq 1$.

Hence the case $\alpha\geqq 1$ and $\gamma>1$ is of interest. The subcases (i) $\alpha>1$ and (ii) $\alpha=1$ are discussed in the subsequent sections. Section 4 is devoted to the estimation of the maxima of square sums and spherical sums, respectively. In the last section we point out that the results can be extended in a natural way to the general multi-parameter case from the two-parameter case, and there is no need to restrict the theorems proved to finite measures.

Throughout the paper, $C, C_1, \ldots$ will denote positive constants, not necessarily the same at different occurrences.

§ 2. An asymptotically optimal inequality in the case $\alpha>1$

Theorem 2 below provides a bound on $E\big(M^{\gamma}(b, m;\ c, n)\big)$ which is asymptotically optimal as $m, n\to\infty$ in the sense that it is of the same order of magnitude as the bound assumed on $E|S(b, m;\ c, n)|^{\gamma}$.

Theorem 2. *Suppose that there exists a non-negative function* $f(b, m;\ c, n)$ *satisfying* (1.1)—(1.2) *such that* (1.3) *holds for all* $b, c\geqq 0$ *and* $m, n\geqq 1$, *where* $\alpha>1$ *and* $\gamma>1$. *Then we have*

$$E\big(M^{\gamma}(b, m;\ c, n)\big) \leqq C_{2,\alpha,\gamma} f^{\alpha}(b, m;\ c, n) \tag{2.1}$$

for all $b, c\geqq 0$ *and* $m, n\geqq 1$.

Although its specific value will have no importance for us, the constant $C_{2,\alpha,\gamma}$ may be taken as

$$C_{2,\alpha,\gamma} = (1-2^{(1-\alpha)/\gamma})^{-2\gamma}. \tag{2.2}$$

Before proving Theorem 2, let us introduce the following "striped" maxima that are the maxima of partial sums taken with respect to only p or q:

$$M_1(b, m;\ c, n) = \max_{1\leqq p\leqq m} |S(b, p;\ c, n)|$$

and

$$M_2(b, m;\ c, n) = \max_{1\leqq q\leqq n} |S(b, m;\ c, q)|,$$

where $b, c\geqq 0$ and $m, n\geqq 1$ are integers.

We need the following auxiliary result in the sequel.

Lemma 1. *Let $\alpha>1$ and $\gamma>1$. Suppose that there exists a non-negative function $f(b, m; c, n)$ satisfying* (1.3) *for all $b, c\geqq 0$ and $m, n\geqq 1$. If* (1.2) *holds, then we have*

$$E\big(M_2^\gamma(b, m;\ c, n)\big) \leqq C_{1,\alpha,\gamma} f^\alpha(b, m;\ c, n) \tag{2.3}$$

for all $b, c\geqq 0$ and $m, n\geqq 1$; here $C_{1,\alpha,\gamma}$ may be taken as

$$C_{1,\alpha,\gamma} = (1-2^{(1-\alpha)/\gamma})^{-\gamma}. \tag{2.4}$$

An analogous result is true for $M_1(b, m; c, n)$ under the assumptions (1.1) and (1.3).

This lemma can be obtained by a simultaneous application to all possible fixed values of $b\geqq 0$ and $m\geqq 1$ of a recent result [6, Theorem 1] in the case when

$$\xi_l = \sum_{k=b+1}^{b+m} \zeta_{kl}, \quad g(c, n) = f(b, m;\ c, n), \quad \text{and} \quad M_{c,n} = M_2(b, m;\ c, n),$$

where $l=c+1, \ldots, c+n$ (the notations are the same as in the cited paper).

Proof of Theorem 2. The proof will be done in a similar way as that of [6, Theorem 1]. We are going to find a constant $C\geqq C_{1,\alpha,\gamma}$, depending only on α and γ, for which the inequality

$$E\big(M^\gamma(b, k;\ c, n)\big) \leqq Cf^\alpha(b, k;\ c, n) \tag{2.5}$$

holds for all $b, c\geqq 0$ and $k, n\geqq 1$.

The proof of (2.5) goes by induction on k. If $k=1$, then (2.5) is a consequence of Lemma 1, since we have

$$M(b, 1;\ c, n) = M_2(b, 1;\ c, n)$$

for all $b, c\geqq 0$ and $n\geqq 1$.

Now assume as induction hypothesis that (2.5) holds for all $k<m$ (and for all $b, c\geqq 0$, $n\geqq 1$) and prove it for $k=m$ (and for all $b, c\geqq 0$, $n\geqq 1$).

If for certain $b, c\geqq 0$ and $m, n\geqq 1$ we have $f(b, m; c, n)=0$, then by (1.1)—(1.2) we also have $f(b, k; c, l)=0$, and hence $S(b, k; c, l)=0$ a.s. for $k=1, 2, \ldots, m$; $l=1, 2, \ldots, n$. Thus $M(b, m; c, n)=0$ a.s., and (2.5) is clearly satisfied.

From now on we assume that $f(b, m; c, n)\neq 0$. Since $f(b, m; c, n)$ is a non-decreasing function in m for any fixed $b, c\geqq 0$ and $n\geqq 1$, there exists an integer h, $1\leqq h\leqq m$, such that

$$f(b, h-1;\ c, n) \leqq \frac{1}{2} f(b, m;\ c, n) < f(b, h;\ c, n), \tag{2.6}$$

where $f(b, h-1; c, n)$ on the left is 0, if $h=1$. Then (1.1) implies

$$f(b+h, m-h;\ c, n) \leqq f(b, m;\ c, n) - f(b, h;\ c, n) < \frac{1}{2} f(b, m;\ c, n). \tag{2.7}$$

Now, for $1 \leqq p < h$ and $1 \leqq q \leqq n$, we have

$$|S(b, p;\ c, q)| \leqq M(b, h-1;\ c, n),$$

and, for $h \leqq p \leqq m$ and $1 \leqq q \leqq n$,

$$|S(b, p;\ c, q)| \leqq M_2(b, h;\ c, n) + M(b+h, m-h;\ c, n).$$

In the last two inequalities we tacitly assume that for either $h=1$ or $h=m$

$$M(b, 0;\ c, n) = M(b+m, 0;\ c, n) = 0.$$

Therefore,

$$M(b, m;\ c, n) \leqq M_2(b, h;\ c, n) + \{M^\gamma(b, h-1;\ c, n) + M^\gamma(b+h, m-h;\ c, n)\}^{1/\gamma}$$

and, by Minkowski's inequality,

$$\text{(2.8)}\quad \{E(M^\gamma(b, m;\ c, n))\}^{1/\gamma} \leqq \{E(M_2^\gamma(b, h;\ c, n))\}^{1/\gamma} + \\ + \{E(M^\gamma(b, h-1;\ c, n)) + E(M^\gamma(b+h, m-h;\ c, n))\}^{1/\gamma}.$$

Applying the induction hypothesis to $M(b, h-1;\ c, n)$, we get that

$$\text{(2.9)}\qquad E(M^\gamma(b, h-1;\ c, n)) \leqq Cf^\alpha(b, h-1;\ c, n) \leqq \frac{C}{2^\alpha} f^\alpha(b, m;\ c, n),$$

the right-most inequality following from (2.6). Applying again the induction hypothesis this time to $M(b+h, m-h;\ c, n)$ and using (2.7), we find that

$$\text{(2.10)}\qquad E(M^\gamma(b+h, m-h;\ c, n)) < \frac{C}{2^\alpha} f^\alpha(b, m;\ c, n).$$

Finally, by (2.3),

$$\text{(2.11)}\qquad E(M_2^\gamma(b, h;\ c, n)) \leqq C_{1,\alpha,\gamma} f^\alpha(b, h;\ c, n) \leqq C_{1,\alpha,\gamma} f^\alpha(b, m;\ c, n).$$

Combining inequalities (2.9)—(2.11) with (2.8), we obtain that

$$\{E(M^\gamma(b, m;\ c, n))\}^{1/\gamma} \leqq (C_{1,\alpha,\gamma}^{1/\gamma} + 2^{(1-\alpha)/\gamma} C^{1/\gamma}) f^{\alpha/\gamma}(b, m;\ c, n).$$

If C is large enough, then it follows that

$$\{E(M^\gamma(b, m;\ c, n))\}^{1/\gamma} \leqq C^{1/\gamma} f^{\alpha/\gamma}(b, m;\ c, n),$$

which proves (2.5) for $k=m$. This completes the induction step and the proof of Theorem 2.

The smallest C satisfying

$$C_{1,\alpha,\gamma}^{1/\gamma} + 2^{(1-\alpha)/\gamma} C^{1/\gamma} \leqq C^{1/\gamma}$$

is given by

$$C = C_{2,\alpha,\gamma} = C_{1,\alpha,\gamma}(1 - 2^{(1-\alpha)/\gamma})^{-\gamma}.$$

By (2.4) this provides (2.2).

Since the stress of this paper is mostly on the method of proving Theorem 2 (Theorem 3 etc. later on), we shall not exhibit the full strength of Theorem 2 and mention only one consequence.

Corollary 1. *Let $\gamma>2$. Suppose that we have*

$$E|S(b, m;\ c, n)|^\gamma \leqq C\left(\sum_{k=b+1}^{b+m}\ \sum_{l=c+1}^{c+n} \sigma_{kl}^2\right)^{\gamma/2}$$

for all $b, c\geqq 0$ and $m, n\geqq 1$, where the σ_{kl}^2 are the finite variances of the rv's ζ_{kl}. Then we have

$$E\big(M^\gamma(b, m;\ c, n)\big) \leqq C(1-2^{(2-\gamma)/2\gamma})^{-2\gamma}\left(\sum_{k=b+1}^{b+m}\ \sum_{l=c+1}^{c+n} \sigma_{kl}^2\right)^{\gamma/2}$$

for all $b, c\geqq 0$ and $m, n\geqq 1$.

The corresponding result for sequences $\{\xi_j\}$ of rv's was established by Erdős [4] and S. B. Stečkin. To be more precise, this result was proved by Erdős for the special case when $\gamma=4$ and $\{\xi_j\}$ is a lacunary sequence of trigonometric functions, while the general case was an oral communication of Stečkin (cf. Gapoškin [5, pp. 29—31]).

§ 3. A generalization of the Rademacher-Menšov inequality in the case $\alpha=1$

Let us proceed to the study of the case $\alpha=1$. Then a factor $(\log 2m)^\gamma$ $(\log 2n)^\gamma$ will occur instead of the constant $C_{2,\alpha,\gamma}$ on the right-hand side of (2.1). Here and in the sequel all logarithms are of base 2.

Theorem 3. *Suppose that there exists a non-negative function $f(b, m;\ c, n)$ satisfying* (1.1)—(1.2) *such that* (1.3) *holds for all $b, c\geqq 0$ and $m, n\geqq 1$, where $\alpha=1$ and $\gamma>1$. Then we have*

$$E\big(M^\gamma(b, m;\ c, n)\big) \leqq (\log 2m)^\gamma(\log 2n)^\gamma f(b, m;\ c, n) \tag{3.1}$$

for all $b, c\geqq 0$ and $m, n\geqq 1$.

This is a special case of the following more general result. Before its formulation, let us introduce two recursive definitions. Let $\varkappa(m)$ and $\lambda(n)$ be positive, non-decreasing functions of the natural numbers m and n, respectively. Set, for $m=1$ and $n=1$,

$$K(1) = \varkappa(1) \quad\text{and}\quad \Lambda(1) = \lambda(1),$$

and set, for $m\geqq 2$ and $n\geqq 2$,

$$K(m) = \varkappa(h)+K(h-1),\quad h = \left[\frac{1}{2}(m+2)\right]; \tag{3.2}$$

$$\Lambda(n) = \lambda(i)+\Lambda(i-1),\qquad i = \left[\frac{1}{2}(n+2)\right];$$

here [.] denotes integral part. It is obvious that both $K(m)$ and $\Lambda(n)$ are positive and non-decreasing functions of $m, n=1, 2, \ldots$. Further, from (3.2) it follows that if $2^p \leqq m < 2^{p+1}$ with $p \geqq 0$, then

$$K(m) \leqq K(2^{p+1}-1) = \sum_{k=0}^{p} \varkappa(2^k); \tag{3.3}$$

similarly, if $2^q \leqq n < 2^{q+1}$ with $q \geqq 0$, then

$$\Lambda(n) \leqq \sum_{l=0}^{q} \lambda(2^l).$$

Theorem 4. *Suppose that there exist positive, non-decreasing functions $\varkappa(m)$ and $\lambda(n)$, and a non-negative function $f(b, m; c, n)$ satisfying* (1.1)—(1.2) *such that*

$$E|S(b, m; c, n)|^\gamma \leqq \varkappa^\gamma(m)\lambda^\gamma(n) f(b, m; c, n) \tag{3.4}$$

for all $b, c \geqq 0$ and $m, n \geqq 1$, where $\gamma \geqq 1$. Let $K(m)$ and $\Lambda(n)$ be defined by (3.2). *Then we have*

$$E\big(M^\gamma(b, m; c, n)\big) \leqq K^\gamma(m)\Lambda^\gamma(n) f(b, m; c, n) \tag{3.5}$$

for all $b, c \geqq 0$ and $m, n \geqq 1$.

We note that if $\varkappa(m) \equiv 1$ and $\lambda(n) \equiv 1$, then $K(m) \leqq \log 2m$ and $\Lambda(n) \leqq \log 2n$. These follow from the inequalities $1 + \log 2(h-1) \leqq \log 2m$ and $1 + \log 2(i-1) \leqq \leqq \log 2n$, which are true owing to $m \geqq 2h-2$ and $n \geqq 2i-2$. Consequently, Theorem 3 is a particular case of Theorem 4.

If the rv's ζ_{kl} are mutually orthogonal, i.e.,

$$E(\zeta_{ij}\zeta_{kl}) = 0 \quad \text{unless} \quad i = k \quad \text{and} \quad j = l, \tag{3.6}$$

and if

$$E\zeta_{kl}^2 = \sigma_{kl}^2, \tag{3.7}$$

then obviously

$$E\big(S^2(b, m; c, n)\big) = \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} \sigma_{kl}^2$$

for all $b, c \geqq 0$ and $m, n \geqq 1$. Hence Theorem 3 implies

Corollary 2. (The two-parameter version of the Rademacher—Menšov inequality) *Under the conditions* (3.6) *and* (3.7) *we have*

$$E\big(M^2(b, m; c, n)\big) \leqq (\log 2m)^2 (\log 2n)^2 \sum_{k=b+1}^{b+m} \sum_{l=c+1}^{c+n} \sigma_{kl}^2$$

for all $b, c \geqq 0$ and $m, n \geqq 1$.

This result was firstly achieved by AGNEW [1]. As for the one-dimensional Rademacher—Menšov inequality see, e.g., RÉVÉSZ [9, p. 83].

If $\varkappa(m)=m^{\beta_1}$ and $\lambda(n)=n^{\beta_2}$ with some positive β_1 and β_2, then by (3.3) we have $K(m)\leqq(2m)^{\beta_1}/(2^{\beta_1}-1)$ and $\Lambda(n)\leqq(2n)^{\beta_2}/(2^{\beta_2}-1)$. Thus in this case we can guarantee again a bound on $E(M^\gamma(b,m;\,c,n))$ of the same order of magnitude as the bound assumed on $E|S(b,m;\,c,n)|^\gamma$.

Corollary 3. *Suppose that there exists a non-negative function $f(b,m;\,c,n)$ satisfying* (1.1)—(1.2) *such that*

$$E|S(b,m;\,c,n)|^\gamma \leqq m^{\gamma\beta_1}n^{\gamma\beta_2}f(b,m;\,c,n)$$

for all $b,c\geqq 0$ and $m,n\geqq 1$, where $\beta_1,\beta_2>0$ and $\gamma\geqq 1$. Then we have

$$E(M^\gamma(b,m;\,c,n)) \leqq \frac{2^{\gamma(\beta_1+\beta_2)}m^{\gamma\beta_1}n^{\gamma\beta_2}}{(2^{\beta_1}-1)^\gamma(2^{\beta_2}-1)^\gamma}f(b,m;\,c,n)$$

for all $b,c\geqq 0$ and $m,n\geqq 1$.

Before proving Theorem 4 we recall the following one-parameter maximal inequality concerning $M_2(b,m;\,c,n)$ defined in § 2.

Lemma 2. *Let $\gamma\geqq 1$. Suppose that there exist positive, non-decreasing functions $\varkappa(m)$ and $\lambda(n)$, and a non-negative function $f(b,m;\,c,n)$ satisfying* (3.4) *for all $b,c\geqq 0$ and $m,n\geqq 1$. If* (1.2) *holds, then we have*

$$E(M_2^\gamma(b,m;\,c,n)) \leqq \varkappa^\gamma(m)\Lambda^\gamma(n)f(b,m;\,c,n) \tag{3.8}$$

for all $b,c\geqq 0$ and $m,n\geqq 1$, where $\Lambda(n)$ is defined by (3.2).

An analogous result is valid for $M_1(b,m;\,c,n)$ under the assumptions (1.1) and (3.4).

Lemma 2 immediately follows from [6, Theorem 4] (cf. the reasoning after Lemma 1).

Proof of Theorem 4. The proof goes by induction on m. If $m=1$, then (3.5) is a consequence of Lemma 2 owing to

$$K(1)=\varkappa(1) \quad\text{and}\quad M(b,1;\,c,n)=M_2(b,1;\,c,n).$$

Let $m>1$ be given and let h be the integral part of $(m+2)/2$. Then $m=2h-1$ or $m=2h-2$. Let $b,c\geqq 0$ and $n\geqq 1$ be arbitrary integers.

In the same way as in the proof of Theorem 2 we arrive at (2.8). Now suppose that the conclusion (3.5) to be proved is true for all $h<m$. Then we obtain

$$E(M^\gamma(b,h-1;\,c,n)) \leqq K^\gamma(h-1)\Lambda^\gamma(n)f(b,h-1;\,c,n)$$

and

$$E(M^\gamma(b+h,m-h;\,c,n)) \leqq K^\gamma(m-h)\Lambda^\gamma(n)f(b+h,m-h;\,c,n) \leqq$$
$$\leqq K^\gamma(h-1)\Lambda^\gamma(n)f(b+h,m-h;\,c,n),$$

since $m \leqq 2h-1$ and the function $K(m)$ is non-decreasing. (In case $m=2$ we have $h=2$, and the second inequality becomes trivial by agreeing that $M(b+2, 0; c, n)=0$ and $K(0)=0$.) Putting these two inequalities together, by (1.1) we find that

$$E(M^\gamma(b, h-1; c, n))+E(M^\gamma(b+h, m-h; c, n)) \leqq \\ \leqq K^\gamma(h-1)A^\gamma(n)f(b, m; c, n). \tag{3.9}$$

By virtue of (3.8)

$$E(M_2^\gamma(b, h; c, n)) \leqq \varkappa^\gamma(h)A^\gamma(n)f(b, h; c, n) \leqq \varkappa^\gamma(h)A^\gamma(n)f(b, m; c, n). \tag{3.10}$$

Collecting inequalities (2.8) and (3.9)—(3.10), we get that

$$\{E(M^\gamma(b, m; c, n))\}^{1/\gamma} \leqq (\varkappa(h)+K(h-1))A(n)f^{1/\gamma}(b, m; c, n).$$

Taking into account the definition (3.2) of $K(m)$, the last inequality gives the wanted (3.5). Thus the proof of Theorem 4 is complete.

§ 4. The maxima of square sums and spherical sums

In this previous sections we established moment inequalities for the maximum of the rectangular sums $S(b, p; c, q)$ as p and q run, independently of each other, over the values $1, 2, \ldots, m$ and $1, 2, \ldots, n$, respectively. The situation becomes simpler if $p=q$ or, more generally, if p and q are connected with each other in a certain way.

Let $Q_1 \subset Q_2 \subset \ldots$ be an arbitrary sequence of finite regions in the positive quadrant R_+^2 of the real plane R^2 such that $\bigcup_{r=1}^{\infty} Q_r$ contains infinitely many points with integer coordinates (but not necessarily coincides with R_+^2). Set

$$T(a, r) = \sum_{(k, l) \in Q_{a+r} \setminus Q_a} \zeta_{kl}$$

and

$$N(a, r) = \max_{1 \leqq s \leqq r} |T(a, s)|,$$

where $a \geqq 0$ and $r \geqq 1$ are integers, $Q_0=\emptyset$.

The assumed bounds on $E|T(a, r)|^\gamma$ will be of the form

$$E|T(a, r)|^\gamma \leqq g^\alpha(a, r), \tag{4.1}$$

where α and γ are given numbers, $\alpha \geqq 1$, $\gamma > 0$, and $g(a, r)$ is a non-negative function with the property

$$g(a, s)+g(a+s, r-s) \leqq g(a, r) \tag{4.2}$$

for all $a \geqq 0$ and $1 \leqq s < r$. Our goal is to deduce an upper bound on $E(N^\gamma(a, r))$.

The one-parameter version of Theorems 1 and 2 reads as follows.

Theorem 5. *Suppose that there exists a non-negative function $g(a, r)$ satisfying* (4.2) *such that* (4.1) *holds for all $a \geqq 0$ and $r \geqq 1$, where either $\alpha \geqq 1$ and $0 < \gamma \leqq 1$ or $\alpha > 1$ and $\gamma > 1$. Then we have*

$$E(N^\gamma(a, r)) \leqq C_{1,\alpha,\gamma} g^\alpha(a, r) \tag{4.3}$$

for all $a \geqq 0$ and $r \geqq 1$.

We remark that the constant $C_{1,\alpha,\gamma}$ in (4.3) may be taken as

$$C_{1,\alpha,\gamma} = \begin{cases} 1 & \text{if} \quad \alpha \geqq 1 \quad \text{and} \quad 0 < \gamma \leqq 1, \\ (1 - 2^{(1-\alpha)/\gamma})^{-\gamma} & \text{if} \quad \alpha > 1 \quad \text{and} \quad \gamma > 1. \end{cases}$$

By setting $\xi_j = \sum_{(k,l) \in Q_j \setminus Q_{j-1}} \zeta_{kl}$ $(j = 1, 2, \ldots)$, Theorem 5 follows immediately from [6, Theorems 1 and 2]. On the other hand, if we apply [6, Theorem 4] to this sequence $\{\xi_j\}$, we get the following one-parameter version of the present Theorem 4.

Theorem 6. *Suppose that there exist a positive, non-decreasing function $\varkappa(r)$, and a non-negative function $g(a, r)$ satisfying* (4.2) *such that*

$$E|T(a, r)|^\gamma \leqq \varkappa^\gamma(r) g(a, r)$$

holds for all $a \geqq 0$ and $r \geqq 1$, where $\gamma \geqq 1$. Let $K(r)$ be defined by (3.2). *Then we have*

$$E(N^\gamma(a, r)) \leqq K^\gamma(r) g(a, r)$$

for all $a \geqq 0$ and $r \geqq 1$.

Let us consider two interesting special cases for the choice of $\{Q_r\}$, which provide (i) the square sums, among others, and (ii) the spherical sums.

Case (i). Let $m = m(r)$ and $n = n(r)$, where

$$1 \leqq m(1) \leqq m(2) \leqq \ldots \quad \text{and} \quad 1 \leqq n(1) \leqq n(2) \leqq \ldots$$

are two sequences of integers such that $\max\{m(r), n(r)\} \to \infty$ as $r \to \infty$, and let $Q_r = \{(k, l)\colon k \leqq m(r) \text{ and } l \leqq n(r)\}$. It is convenient to put $m(0) = n(0) = 0$ and $Q_0 = \emptyset$. Now

$$T(a, r) = \left\{ \sum_{k=1}^{m(a+r)} \sum_{l=1}^{n(a+r)} - \sum_{k=1}^{m(a)} \sum_{l=1}^{n(a)} \right\} \zeta_{kl}.$$

In particular, if $m(r) = n(r) = r$, then the $T(0, r) = S(0, r; 0, r)$ give back the square sums. The case $m(r) = n(r) = 2^r$ is also of interest.

We mention that if $f(b, m; c, n)$ is a non-negative function satisfying (1.1)—(1.2), then $g(a, r)$ defined by

$$g(a, r) = f(0, m(a+r);\ 0, n(a+r)) - f(0, m(a);\ 0, n(a))$$

is also non-negative and satisfies (4.2).

It is worth stating Theorem 6 explicitly in the special case of mutually orthogonal ζ_{kl}. Then $\gamma=2$, $\varkappa(r)\equiv 1$, and with $\sigma_{kl}^2=E(\zeta_{kl}^2)$ we have

$$E\big(T^2(a,r)\big)=\left\{\sum_{k=1}^{m(a+r)}\sum_{l=1}^{n(a+r)}-\sum_{k=1}^{m(a)}\sum_{l=1}^{n(a)}\right\}\sigma_{kl}^2$$

for all $a\geqq 0$ and $r\geqq 1$.

Corollary 4. *Let $\{m(r)\}$ and $\{n(r)\}$ be two non-decreasing sequences of positive integers. Under the conditions* (3.6) *and* (3.7) *we have*

$$E\big(N^2(a,r)\big)\leqq(\log 2r)^2\left\{\sum_{k=1}^{m(a+r)}\sum_{l=1}^{n(a+r)}-\sum_{k=1}^{m(a)}\sum_{l=1}^{n(a)}\right\}\sigma_{kl}^2$$

for all $a\geqq 0$ and $r\geqq 1$.

Case (ii). The spherical sums are defined with the aid of $Q_r=\{(k,l)\colon k^2+l^2\leqq r\}$ $(r=1,2,\ldots;\ Q_1=\emptyset)$, i.e., now

$$T(a,r)=\sum_{a<k^2+l^2\leqq a+r}\zeta_{kl}.$$

The case of orthogonal ζ_{kl} is again of interest in itself.

Corollary 5. *Under the conditions* (3.6) *and* (3.7) *we have*

$$E\big(N^2(a,r)\big)\leqq(\log 2r)^2\sum_{a<k^2+l^2\leqq a+r}\sigma_{kl}^2$$

for all $a\geqq 1$ and $r\geqq 1$.

Corollaries 4 and 5 were proved earlier in [7, Corollary 3 and Theorem 4].

§ 5. Generalizations to multiparameter case

Let Z^d denote the set of all d-tuples of non-negative integers, and let Z_+^d denote the set of all d-tuples of positive integers, where $d\geqq 1$ is a fixed integer. The points in Z^d are denoted by $\mathbf{k}, \mathbf{m}$ etc., or sometimes, when necessary, more explicitly by $(k_1,k_2,\ldots,k_d)$, $(m_1,m_2,\ldots,m_d)$ etc. Two d-tuples $\mathbf{k}$ and $\mathbf{m}$ are said to be distinct if for at least one j we have $k_j\neq m_j$ $(1\leqq j\leqq d)$. Z^d is partially ordered by agreeing that $\mathbf{k}\leqq\mathbf{m}$ iff $k_j\leqq m_j$ for each j, $1\leqq j\leqq d$. We write $\mathbf{0}$ and $\mathbf{1}$ respectively for the points $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$ in Z^d.

Let $\{\zeta_{\mathbf{k}}\}=\{\zeta_{\mathbf{k}}\colon \mathbf{k}\in Z_+^d\}$ be a random field, i.e. a collection of rv's indexed by the set Z_+^d. Put

$$S(\mathbf{b},\mathbf{m})=\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}}\zeta_{\mathbf{k}}=\sum_{k_1=b_1+1}^{b_1+m_1}\cdots\sum_{k_d=b_d+1}^{b_d+m_d}\zeta_{k_1,\ldots,k_d}$$

and

$$M(\mathbf{b},\mathbf{m})=\max_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}}|S(\mathbf{b},\mathbf{k})|=\max_{1\leqq k_1\leqq m_1}\cdots\max_{1\leqq k_d\leqq m_d}|S(\mathbf{b},\mathbf{k})|,$$

where $\mathbf{b}\in Z^d$, $\mathbf{m}\in Z_+^d$, and $\mathbf{b}+\mathbf{1}$, $\mathbf{b}+\mathbf{m}$ are the usual coordinatewise sums.

To formulate the generalizations of the above Theorems 1—6, let $f(\mathbf{b},\mathbf{m})$ denote a non-negative function depending on the joint df of $\{\zeta_{\mathbf{k}}:\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}\}$ with the following property. Set, for $1\leqq j\leqq d$,

$$\mathbf{b}_j=(b_1,\dots,b_{j-1},b_{j+1},\dots,b_d)\in Z^{d-1}$$

and

$$\mathbf{m}_j=(m_1,\dots,m_{j-1},m_{j+1},\dots,m_d)\in Z_+^{d-1},$$

where $\mathbf{b}=(b_1,b_2,\dots,b_d)\in Z^d$ and $\mathbf{m}=(m_1,m_2,\dots,m_d)\in Z_+^d$. We require that the inequality

$$f(\mathbf{b}_j,\mathbf{m}_j;b_j,h_j)+f(\mathbf{b}_j,\mathbf{m}_j;b_j+h_j,m_j-h_j)\leqq f(\mathbf{b}_j,\mathbf{m}_j;b_j,m_j)=f(\mathbf{b},\mathbf{m}) \tag{5.1}$$

holds true for all $\mathbf{b}\in Z^d$, $\mathbf{m}\in Z_+^d$, $1\leqq h_j<m_j$, and $1\leqq j\leqq d$.

Inequality (5.1) expresses that $f(\mathbf{b},\mathbf{m})$ as a function of the interval (b_j+1,b_j+m_j) is superadditive for any fixed values of $b_1,m_1,\dots,b_{j-1},m_{j-1},b_{j+1},m_{j+1},\dots,b_d,m_d$. Examples are $f(\mathbf{b},\mathbf{m})=m_1^{\beta_1}m_2^{\beta_2}\dots m_d^{\beta_d}$ with $\beta_j\geqq 1$ for each j, $1\leqq j\leqq d$; or $f(\mathbf{b},\mathbf{m})=\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}}\sigma_{\mathbf{k}}^2$, in the latter case assuming the existence of the finite variances $\sigma_{\mathbf{k}}^2$ of the rv's $\zeta_{\mathbf{k}}$.

Theorem 7. *Suppose that there exists a non-negative function $f(\mathbf{b},\mathbf{m})$ satisfying* (5.1) *such that*

$$E|S(\mathbf{b},\mathbf{m})|^\gamma\leqq f^\alpha(\mathbf{b},\mathbf{m})$$

holds for all $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$, where either $\alpha\geqq 1$ and $0<\gamma\leqq 1$ or $\alpha>1$ and $\gamma>1$. Then we have

$$E\big(M^\gamma(\mathbf{b},\mathbf{m})\big)\leqq C_{d,\alpha,\gamma}f^\alpha(\mathbf{b},\mathbf{m})$$

for all $\mathbf{b}\in Z^d$ and $\mathbf{m}\in Z_+^d$.

Here the constant $C_{d,\alpha,\gamma}$ may be chosen as follows: $C_{d,\alpha,\gamma}=1$ if $\alpha\geqq 1$ and $0<\gamma\leqq 1$, and

$$C_{d,\alpha,\gamma}=C_{1,\alpha,\gamma}^d=(1-2^{(1-\alpha)/\gamma})^{-d\gamma}$$

if $\alpha>1$ and $\gamma>1$.

In connection with Theorem 7 we note that Bickel and Wichura [2] proved a fine but not comparable result, providing a multiparameter extension of Billingsley's main fluctuation inequality [3, Theorem 12.5]. Roughly speaking, they obtain an asymptotically optimal inequality on $P\{M(\mathbf{b},\mathbf{m})\geqq\lambda\}$ in terms of assumed bounds on $P\{|S(\mathbf{b},\mathbf{m})|\geqq\lambda\}$, where $\mathbf{b}\in Z^d$, $\mathbf{m}\in Z_+^d$, and λ is a positive number.

For each j, $1\leqq j\leqq d$, let $\lambda_j(m_j)$ be a positive and non-decreasing function of the natural number m_j. Define $\Lambda_j(m_j)$ by the recurrence relation (3.2), that is, for $m_j=1$ set $\Lambda_j(1)=\lambda_j(1)$, and for $m_j\geqq 2$ set

$$\Lambda_j(m_j)=\lambda_j(h_j)+\Lambda_j(h_j-1),\quad\text{where}\quad h_j=\left[\frac{1}{2}(m_j+2)\right]. \tag{5.2}$$

Theorem 8. *Suppose that there exist positive, non-decreasing functions* $\lambda_j(m_j)$ *for* $j=1, 2, \ldots, d$, *and a non-negative function* $f(\mathbf{b}, \mathbf{m})$ *satisfying* (5.1) *such that*

$$E|S(\mathbf{b}, \mathbf{m})|^\gamma \leqq \prod_{j=1}^{d} \lambda_j^\gamma(m_j) f(\mathbf{b}, \mathbf{m}) \tag{5.3}$$

holds for all $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$, *where* $\gamma\geqq 1$. *Let* $\Lambda_j(m_j)$ *be defined by* (5.2) *for* $j=1, 2, \ldots, d$. *Then we have*

$$E(M^\gamma(\mathbf{b}, \mathbf{m})) \leqq \prod_{j=1}^{d} \Lambda_j^\gamma(m_j) f(\mathbf{b}, \mathbf{m}) \tag{5.4}$$

for all $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$.

The proof of Theorems 7 and 8 may be carried out by induction on d in the same manner as we did it from $d=1$ to $d=2$ in the case of Theorems 2 and 4. The simplest case $d=1$ was proved in [6].

As is well-known, the random field $\{\zeta_\mathbf{k}\}$ is said to be orthogonal if

$$E(\zeta_\mathbf{k}\zeta_\mathbf{l}) = 0 \quad \text{if} \quad \mathbf{k} \neq \mathbf{l}. \tag{5.5}$$

Setting

$$E(\zeta_\mathbf{k}^2) = \sigma_\mathbf{k}^2, \tag{5.6}$$

for orthogonal $\zeta_\mathbf{k}$ we obviously have

$$E(S^2(\mathbf{b}, \mathbf{m})) = \sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} \sigma_\mathbf{k}^2.$$

This is a particular case of the condition (5.3) with $\gamma=2$, $\lambda_j(m_j)\equiv 1$ for each j, $1\leqq j\leqq d$, and $f(\mathbf{b}, \mathbf{m})=\sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} \sigma_\mathbf{k}^2$ (the latter is even additive). Now $\Lambda_j(m_j)=\log 2m_j$ for each j and Theorem 8 provides the following

Corollary 6. (The d-parameter version of the Rademacher—Menšov inequality) *Under the conditions* (5.5) *and* (5.6) *we have*

$$E(M^2(\mathbf{b}, \mathbf{m})) \leqq \prod_{j=1}^{d} (\log 2m_j)^2 \sum_{\mathbf{b}+\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{b}+\mathbf{m}} \sigma_\mathbf{k}^2$$

for all $\mathbf{b}\in Z^d$ *and* $\mathbf{m}\in Z_+^d$.

Similar generalizations from 2 to d of Theorems 5 and 6 are valid, too. Instead of stating them explicitly, we formulate a useful consequence for orthogonal $\zeta_\mathbf{k}$. Let $Q_1 Q_2 \ldots$ be an arbitrary sequence of finite regions in Z_+^d such that $\bigcup_{r=1}^{\infty} Q_r$ is not bounded (but may not coincide with Z_+^d). Set

$$T(a, r) = \sum_{\mathbf{k}\in Q_{a+r}\setminus Q_a} \zeta_\mathbf{k}$$

and

$$N(a, r) = \max_{1\leqq s\leqq r} |T(a, s)|,$$

where $a\geqq 0$ and $r\geqq 1$ are integers, $Q_0=\emptyset$.

Corollary 7. *Under the conditions* (5.5) *and* (5.6) *we have*

$$E\big(N^2(a, r)\big) \leqq (\log 2r)^2 \sum_{\mathbf{k}\in Q_{a+r}\setminus Q_a} \sigma_{\mathbf{k}}^2 \tag{5.7}$$

for all $a \geqq 0$ *and* $r \geqq 1$.

We note that in the more general setting when the coordinates m_j of $\mathbf{m}\in Z_+^d$, $1\leqq j\leqq d$, depend on an e-dimensional parameter $\mathbf{r}=(r_1, r_2, \ldots, r_e)\in Z_+^e$, where $1\leqq e<d$, the following result can be achieved for orthogonal $\zeta_{\mathbf{k}}$. Let $\{Q_{\mathbf{r}}: \mathbf{r}\in Z^e\}$ be an arbitrary collection of finite regions in Z_+^d such that $Q_{\mathbf{0}}=\emptyset$, $Q_{\mathbf{s}}\subset Q_{\mathbf{r}}$ if $\mathbf{s}\leqq\mathbf{r}$, and $\bigcup_{\mathbf{r}\geqq 0} Q_{\mathbf{r}}$ is not bounded in Z_+^d, where $\mathbf{r}, \mathbf{s}\in Z^e$. Set

$$T(\mathbf{a}, \mathbf{r}) = \sum_{\mathbf{k}\in Q_{\mathbf{a}+\mathbf{r}}\setminus Q_{\mathbf{a}}} \zeta_{\mathbf{k}}$$

and

$$N(\mathbf{a}, \mathbf{r}) = \max_{1\leqq \mathbf{s}\leqq \mathbf{r}} |T(\mathbf{a}, \mathbf{s})| = \max_{1\leqq s_1\leqq r_1} \ldots \max_{1\leqq s_e\leqq r_e} |T(\mathbf{a}, \mathbf{s})|,$$

where $\mathbf{a}\in Z^e$ and $\mathbf{r}\in Z_+^e$. Then, under (5.5) and (5.6), we have

$$E\big(N^2(\mathbf{a}, \mathbf{r})\big) \leqq \prod_{i=1}^{e} (\log 2r_i)^2 \sum_{\mathbf{k}\in Q_{\mathbf{a}+\mathbf{r}}\setminus Q_{\mathbf{a}}} \sigma_{\mathbf{k}}^2 \tag{5.8}$$

for all $\mathbf{a}\in Z^e$ and $\mathbf{r}\in Z_+^e$.

In case $e=1$, (5.8) reduces to (5.7).

Finally, we mention that viewing our proofs, it is striking that we use no full power of a probability space. In fact, Minkowski's inequality was applied only, which is available in any measure space $(X, \mathcal{A}, \mu)$. Hence our theorems are true in $(X, \mathcal{A}, \mu)$, too, taking integrals over X with respect to μ in place of the expectations on the left-hand sides of the corresponding inequalities.

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